5.5 Substitution technique

Recall: $F(x) = \int f(x)\,dx$

if $F'(x) = f(x)$, and $F(x)$

is called the antiderivative of $f(x)$

In general, we write $\int f(x)\,dx = F(x) + C$

(general antiderivative)

For instance $\int 3x^2\,dx = x^3 + C$

Here, recall: $[f(u)]' = f'(u) \cdot u'$

$= f'(u)\,du$

$\Longrightarrow \int f'(u)\,du = f(u) + C$

In other words $\boxed{\int f(u)\,du = F(u) + C}$

where $u = u(x)$

Examples: Find the indefinite integrals:

#1 $\int (4+3x)^4\, 3\,dx$

Substitution: $u = 4+3x \longrightarrow \frac{du}{dx} = 3$

$\longrightarrow du = 3\,dx$

$\int u^4 (du) \sim \int x^4\,dx$

Recall: $\int x^n\,dx = \begin{cases} \frac{1}{n+1}x^{n+1} + C \\ \ln|x| + C, \; n=-1 \end{cases}$

$\longrightarrow \frac{1}{5}u^5 + C$

Return to the original variable

So since $u = 4+3x$, we have

$\int (4+3x)^4\, 3\,dx = \frac{1}{5}(4+3x)^5 + C$

Check: $\left[\frac{1}{5}(3x+4)^5\right]' = \frac{5}{5}(3x+4)^4(3) = (3x+4)^4(3)$ ✓

#2 $\int \sqrt[3]{(4-5x^2)}\,(-10x)\,dx$

Let $u = 4-5x^2$

$du = -10x\,dx$

(expression of u-only) $\Rightarrow \int \sqrt[3]{u}\,(du)$

$\Rightarrow \int u^{\frac{1}{3}}\,du = \frac{3}{4}u^{\frac{4}{3}} + C$

(expression of x-only) $\boxed{\frac{3}{4}\left[4-5x^2\right]^{\frac{4}{3}} + C}$

or $\frac{3}{4}\sqrt[3]{(4-5x^2)^4} + C$

#3 $\int x^2 (x^3+2)^9 \, dx$

$u = x^3 + 2 \longrightarrow du = 3x^2 dx$

Observe $\int x^2 (x^3+2)^9 \, dx = \int (x^3+2)^9 (x^2 \, dx)$

and $du = 3x^2 dx \longrightarrow \frac{1}{3} du = x^2 dx$

So we have: $\int u^9 \left(\frac{1}{3} du\right) \sim \int \frac{1}{3} x^9 dx$

$= \int \frac{1}{3} u^9 \, du$

Recall: $\int k f(x) \, dx = k \int f(x) \, dx$

$= \frac{1}{3}\left[\frac{1}{10} u^{10}\right] + C$

$= \frac{1}{30} u^{10} + C = \boxed{\frac{1}{30}\left[x^3+2\right]^{10} + C}$

#6 $\displaystyle\int \frac{x}{(8-2x^2)^3}\,dx = \int \frac{1}{(8-2x^2)^3}(x\,dx)$

Let $u = 8-2x^2 \longrightarrow du = -4x\,dx$

$\longrightarrow -\frac{1}{4}du = x\,dx$

So $\displaystyle\int \frac{1}{u^3}\left(-\frac{1}{4}du\right) = -\frac{1}{4}\int \frac{1}{u^3}du \sim -\frac{1}{4}\int \frac{1}{x^3}dx$

$\displaystyle = -\frac{1}{4}\int u^{-3}du$

$\displaystyle = -\frac{1}{4}\left(\frac{1}{-2}\right)u^{-2} + C$

$\displaystyle = \frac{1}{8}u^{-2} + C$

$\displaystyle \boxed{= +\frac{1}{8}\left(8-2x^2\right)^{-2} + C}$

OR $\displaystyle \boxed{\frac{+1}{8(8-2x^2)^2} + C}$

#8 $\displaystyle\int \frac{t-2t^4}{\sqrt{t}}\,dt$ (substitution is not needed!)

$$= \int\left(\frac{t}{\sqrt{t}} - \frac{2t^4}{\sqrt{t}}\right)dt$$

$$= \int\left(t\cdot t^{-1/2} - 2t^4 t^{-1/2}\right)dt$$

$$= \int\left(t^{1/2} - 2t^{7/2}\right)dt$$

$$= \frac{2}{3}t^{3/2} - 2\left(\frac{2}{9}\right)t^{9/2} + C$$

$$\boxed{= \frac{2}{3}\sqrt{t^3} - \frac{4}{9}\sqrt{t^9} + C}$$

#10 $\displaystyle\int \frac{1}{\theta^2}\cos\left(\frac{1}{\theta}\right)d\theta$

$u = \frac{1}{\theta} \longrightarrow du = -\frac{1}{\theta^2}\,d\theta$

$\hookrightarrow u = \theta^{-1} \longrightarrow \frac{du}{d\theta} = -1\theta^{-2} \longrightarrow du = -\theta^{-2}\,d\theta$

So $\displaystyle\int \cos\left(\frac{1}{\theta}\right)\left(\frac{1}{\theta^2}\,d\theta\right) = \int \cos(u)(-du)$

$$= -\int \cos u\,du$$

$$= -\sin u + C$$

$$\boxed{= -\sin\left(\frac{1}{\theta}\right) + C}$$

#12 $\int \sec(8-x)\tan(8-x)\,dx$

$u = 8 - x \longrightarrow du = -dx$ ie $-du = dx$

We have $\int \sec u \tan u\,(-du)$

$= -\int \sec u \tan u\,du$

$= -\sec u + C$ since $(\sec x)' = \sec x \tan x$

$= -\sec(8-x) + C$

#13 $\int \sqrt{\tan(6x)}\,\sec^2(6x)\,dx$

Let $u = \tan(6x)$

$du = \sec^2(6x)\cdot(6)\,dx$

$\longrightarrow \frac{1}{6}du = \sec^2(6x)\,dx$

So we have $\int \sqrt{u}\left(\frac{1}{6}du\right) = \frac{1}{6}\int \sqrt{u}\,du$

$= \frac{1}{6}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{6}\left(\frac{2}{3}\right)u^{3/2} + C$

$= \frac{1}{9}u^{3/2} + C$

OR $\frac{1}{9}\sqrt{\tan^3(6x)} + C = \frac{1}{9}\left[\tan(6x)\right]^{3/2} + C$

#15 $\int \frac{e^{3x}+3e^{x}+2}{e^{x}}\,dx$

$= \int \left(\frac{e^{3x}}{e^{x}} + \frac{3e^{x}}{e^{x}} + \frac{2}{e^{x}}\right) dx$

$= \int \left(e^{3x-x} + 3e^{x-x} + 2e^{-x}\right) dx$

$= \int \left(e^{2x} + 3e^{0} + 2e^{-x}\right) dx$

$= \int e^{2x}\,dx + 3\int 1\,dx + 2\int e^{-x}\,dx$

$\int e^{u}\,du = e^{u} + C$

$\int e^{2x}\,dx$. let $u = 2x \rightarrow du = 2\,dx$

$\frac{1}{2}du = dx$

$\int e^{u}\left(\frac{1}{2}du\right) = \frac{1}{2}\int e^{u}\,du = \frac{1}{2}e^{2x} + C$

$\Rightarrow \boxed{\frac{1}{2}e^{2x} + 3x + 2\left(-e^{-x}\right) + C}$

or $\frac{1}{2}e^{2x} + 3x - \frac{2}{e^{x}} + C$

17 $\displaystyle\int \frac{x^3-3x^2+9}{x-3}\,dx \longrightarrow \frac{P(x)}{D(x)} \longrightarrow$

Long division first.

$Q(x) + \frac{R(x)}{D(x)}$ ← remainder

↑ Quotient

Long division

$x^3-3x^2+9 = x^3-3x^2+0x+9$

D(x) ↓

$$\begin{array}{r|l} & x^2 \leftarrow Q(x) \\ \hline x-3 & x^3-3x^2+0x+9 \\ & -x^3 \pm 3x^2 \\ \hline & +0x+9 \leftarrow R(x) \end{array}$$

By a long division, $\frac{x^3-3x^2+9}{x-3} = \frac{9}{x-3} + x^2$

So $\displaystyle\int \frac{x^3-3x^2+9}{x-3}\,dx = \int\left(\frac{9}{x-3}+x^2\right)dx$

$\displaystyle = \underbrace{\int \frac{9}{x-3}\,dx}_{a} + \underbrace{\int x^2\,dx}_{b}$

Now $\displaystyle\int \frac{9}{x-3}\,dx$ (a)

$u = x-3 \to du = +1\,dx$

So, $\displaystyle\int \frac{9}{u}\,du = 9\int \frac{1}{u}\,du = 9\int u^{-1}\,du$

$\Longrightarrow 9\ln|u| + C \to 9\ln|x-3| + C$ (a)

Also, we have, $\displaystyle\int x^2\,dx = \frac{1}{3}x^3 + C$ (b)

Hence Ans: $\boxed{9\ln|x-3| + \frac{1}{3}x^3 + C}$